MATH4250 Game Theory
Exercise 1

Assignment 1: 1,2,4,5,6,8,9 (Due: 29 Jan 2019 (Tue))

1. Let $\oplus$ denotes the nim-sum.
   (a) Find $27 \oplus 17$
   (b) Find x if $x \oplus 38 = 25$.
   (c) Prove that if $x \oplus y \oplus z = 0$, then $x = y \oplus z$.
2. Let $\oplus$ denotes the nim-sum.
   (a) Find $29 \oplus 20 \oplus 15$.
   (b) Find all winning moves of the game of nim from the position $(29, 20, 15)$.
3. Find all winning moves in the game of nim,
   (a) with three piles of 12, 19, and 27 chips.
   (b) with four piles of 13, 17, 19, and 23 chips.
4. Consider the subtraction game with $S = \{1, 3, 4, 5\}$.
   (a) Find the set of P-positions of the game.
   (b) Prove your assertion in (a).
   (c) Let $g(x)$ be the Sprague-Grundy function of the game. Find $g(4)$, $g(18)$ and $g(29)$.
5. Let $g(x)$ be the Sprague-Grundy function of the subtraction game with subtraction set $S = \{1, 2, 6\}$.
   (a) Find $g(4)$, $g(6)$ and $g(100)$.
   (b) Find all winning moves for the first player if initially there are 100 chips.
   (c) Find the set of P-positions of the game and prove your assertion.
6. In a 2-pile take-away game, there are 2 piles of chips. In each turn, a player may either remove any number of chips from one of the piles, or remove the same number of chips from both piles. The player removing the last chip wins.
   (a) Find all winning moves for the starting positions $(6, 9)$, $(11, 15)$ and $(13, 20)$.
   (b) Find (x, y) if (x, y) is a P-position and
      (i) $x = 100$
      (ii) $x = 500$
      (iii) $x - y = 999$

7. In a staircase nim game there are 5 piles of coins. Two players take turns moving. A move consists of removing any number of coins from the first pile or moving any number of coins from the k-th pile to the $k-1$-th pile for $k = 2, 3, 4, 5$. The player who takes the last coin wins. Let $(x_1, x_2, \cdots, x_5)$ denotes the position with x_i coins in the i-th pile.
   (a) Prove that $(x_1, x_2, \cdots, x_5)$ is a P-position if and only if (x_1, x_3, x_5) is a P-position in the ordinary nim.
   (b) Determine all winning moves from the initial position $(4, 6, 9, 11, 14)$.

8. Consider the following 3 games with normal play rule.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6\}$
Game 3:	When there are n chips remaining, a player can only remove 1 chip if n is odd and can remove any even number of chips if n is even.

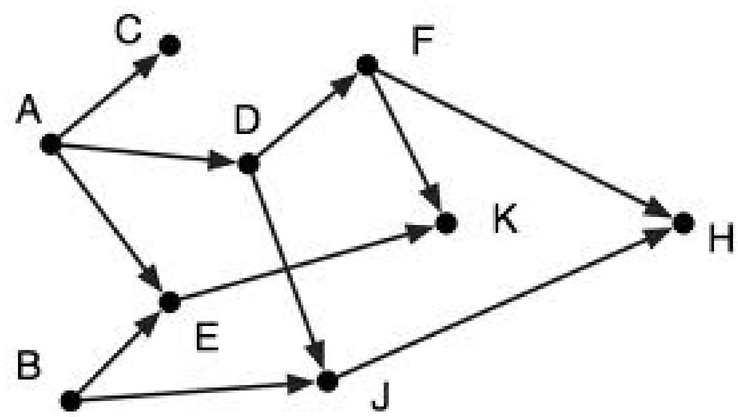
   Let g_1, g_2, g_3 be the the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the 3 games and g be the Sprague-Grundy function of G.
   (a) Find $g_1(14), g_2(17), g_3(24)$.
   (b) Find $g(14, 17, 24)$.
   (c) Find all winning moves of G from the position $(14, 17, 24)$.

9. Consider the following 3 games.

Game 1:	1-pile nim
Game 2:	Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6, 7\}$
Game 3:	When there are n chips remaining, a player can remove any **odd** number of chips if n is odd and can remove 1 or 2 chips if n is even.

   Let g_1, g_2, g_3 be the Sprague-Grundy functions of the 3 games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.
   (a) Find $g_1(7), g_2(14), g_3(18)$.
   (b) Find $g(7, 14, 18)$.
   (c) Find all winning moves of G from the position $(7, 14, 18)$.

10. Consider the game associated with the following directed graph



(a) Copy the graph and write down the value of the Sprague-Grundy function of each vertex.

(b) Write down all vertices which are at P-position but not at terminal position.

(c) Consider the sum of three copies of the given graph game.

(i) Find $g(A, B, E)$ where g is the Sprague-Grundy function.

(ii) Find all winning moves from (A, B, E).

11. Let $g(x)$ be the Sprague-Grundy function of the take-and-break game.

(a) Write down $g(10)$, $g(11)$, $g(12)$.

(b) Find all winning moves from $(10, 11, 12)$

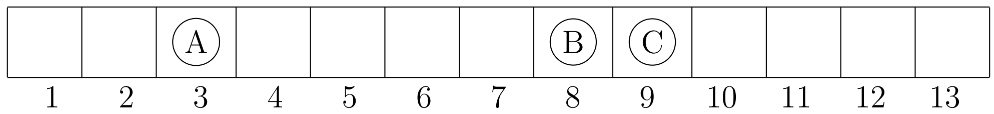
12. For real number $x \in \mathbb{R}$, denote by $\lfloor x \rfloor$ the largest integer such that $\lfloor x \rfloor \leq x$ and $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Let $\alpha, \beta > 1$ be irrational real number such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. Let k be a positive integer.

(a) Prove there exists positive integer n such that $\lfloor n\alpha \rfloor = k$ if and only if $\left\{ \dfrac{k}{\alpha} \right\} > \dfrac{1}{\beta}$.

(b) Prove that exactly one of the following statements holds:

- there exists positive integer n such that $\lfloor n\alpha \rfloor = k$;
- there exists positive integer n such that $\lfloor n\beta \rfloor = k$.

13. A game is played on a game board consisting of a line of squares labeled $1, 2, 3, \ldots$ from left to right. Three coins A, B, C are placed on the squares and at any time each square can be occupied by at most one coin. A move consists of taking one of the coins and moving it to a square with a small number so that coin A occupies a square with a number smaller than coin B and coin B occupies a square with a number smaller than coin C. The game ends when there is no possible move, that is coins A, B, C occupy at square number $1, 2, 3$ respectively, and the player who makes the last move wins. Let (x, y, z), where $1 \leq x < y < z$, be the position of the game that coins A, B, C are at squares labeled x, y, z respectively. The position $(3, 8, 9)$ is shown below.



Examples of legal moves from position $(3, 8, 9)$ are $(1, 8, 9)$, $(3, 4, 9)$ and $(3, 5, 9)$. One cannot move coin C from position $(3, 8, 9)$. Define

$$g(x, y, z) = (x - 1) \oplus (z - y - 1), \text{ for } 1 \leq x < y < z$$

where $\oplus$ denotes nim-sum.

(a) Prove that $g(x, y, z)$ is the Sprague-Grundy function of the game. (All properties of $\oplus$ can be used without proof.)

(b) Find all winning moves from the positions $(6, 13, 25)$ and $(23, 56, 63)$.